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SHORTER COMMUNICATIONS

EFFECTS OF VARIABLE PHYSICAL PROPERTIES IN LAMINAR FLOW OF PSEUDOPLASTIC FLUIDS

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NOMENCLATURE

a, coefficient defined by equation (9);
B, coefficient defined by equation (3);
$\hat{c}_p$, fluid specific heat;
D, tube diameter;
g, acceleration of gravity;
Gr, $= \rho^2 \beta g (T_w - T_0) R^{2n+1} v_m^{2-2n}/m^2$, Grashof number;
G, mass flow rate;
Gz, $= G\hat{c}_p/(kL)$, Graetz number;
h_{av}, average heat-transfer coefficient;
k, thermal conductivity;
L, tube length;
m, fluid consistency;
M, coefficient defined by equation (5);
n, index of pseudoplasticity, by equation (2);
Nu_{av}, $= h_{av} D/k$, average Nusselt number;
Nu_m, $= (1/\eta)\ln(1/\theta_b)$, mean Nusselt number;
p, pressure;
P, $= (p + \rho_r g z)/(\rho_r v_m^2 Pr)$, dimensionless pressure;
Pr, $= m\hat{c}_p (v_m/R)^{n-1}/k$, Prandtl number;
r, radial coordinate;
R, tube radius;
Re, $= \rho v_m^{2-n} R^n/m$, Reynolds number;
T, temperature;
T_r, reference temperature;
V, $= V_z/v_m$, dimensionless axial velocity;
v_m, mean axial velocity;
V_r, radial velocity;
W, $= V_r Re Pr/v_m$, dimensionless radial velocity;
z, axial coordinate.

Greek symbols

β, coefficient of thermal cubic expansion;
Γ_{zr}, shear stress, equation (2);
Δw, $= (3n+1)/4n$;
η, $= z/(R Re Pr)$, dimensionless axial coordinate;
θ, $= (T - T_w)/(T_0 - T_w)$, dimensionless temperature;
ξ, $= r/R$, dimensionless radial coordinate;
ρ, fluid density.

Subscripts

b, evaluated at bulk temperature;
0, evaluated at inlet temperature;
w, evaluated at wall temperature;
r, evaluated at reference temperature.

INTRODUCTION

THE HEATING of pseudoplastic fluids in laminar flow has been the subject of many studies, due to the great interest in these fluids evinced by industry.

Some workers have been concerned with the heating in a vertical tube with constant wall temperature. The effects of variable consistency in forced laminar convection and fully developed axial velocity at the inlet have been studied by Christiansen *et al.*, who examined pressure drop [1] and average heat transfer, during heating [2] and cooling [3]. McKillop *et al.* [4] have studied the influence of variable consistency on pressure drop and local heat transfer in forced laminar convection with flat axial velocity profile at the inlet. Marner *et al.* have pointed out the importance of natural convection during the heating of Newtonian [5] and pseudoplastic fluids [6].

As observed by Metzner [7], it is extremely important to examine the simultaneous variations of density and consistency. The two combined effects have been considered by Mitsuishi *et al.* in a vertical annulus [8, 9] and in a vertical tube with constant wall heat flux [10].

The purpose of the present paper is to examine the heating of pseudoplastic fluids in mixed laminar convection with variable consistency in a vertical tube with constant wall temperature and fully developed axial velocity at the inlet.

EQUATIONS

Fluid motion is represented by the usual equations of momentum, energy, continuity and the continuity integral [11], under the common hypotheses [6]. Momentum equation is

$$\rho\left\{V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z}\right\} = -\frac{\mathrm{d}p}{\mathrm{d}z} - \rho g + \frac{1}{r}\frac{\partial}{\partial r}\{r\Gamma_{zr}\}. \quad (1)$$

For pseudoplastic fluids, the shear stress, Γ_{zr}, is given by

$$\Gamma_{zr} = m\left|\frac{\partial V_z}{\partial r}\right|^{n-1} \frac{\partial V_z}{\partial r}; \quad (2)$$

the consistency, m, is

$$m = m_r \exp\left\{B\left(\frac{1}{T} - \frac{1}{T_r}\right)n\right\} \quad (3)$$

where m_r is evaluated at the reference temperature T_r.

The coefficient B in equation (3) is determined by the values of consistency at the wall and the inlet; equation (4) becomes

$$\frac{m}{m_r} = \exp\left\{\frac{(\theta_r - \theta)\ln(m_w/m_0)}{\{1 + \theta_r(T_0/T_w - 1)\}\{T_w/T_0 + \theta(1 - T_w/T_0)\}}\right\} \quad (4)$$

Equation (4) has a general form and can be reduced to the expressions used employing different reference temperatures to evaluate the physical properties of the fluid. When the reference temperature has the same value as the inlet, equation (4) reduces to that of Rosenberg *et al.* [12].

In the present work, the reference wall temperature will be used as this is usual.

The parameter M will be employed to indicate the variations of consistency

$$M = \ln(m_w/m_0)/n. \quad (5)$$

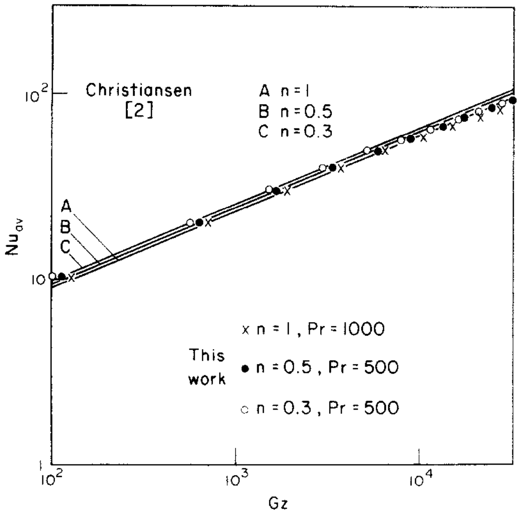

Fig. 1. Forced convection with variable consistency ($M = -2$, $T_w/T_0 = 1.2$).

The density is a linear function of temperature. Initial and boundary conditions refer to a fully developed flow and constant wall temperature.

The dimensionless momentum equation is

$$\frac{1}{Pr}\left\{W\frac{\partial V}{\partial \xi}+V\frac{\partial V}{\partial \eta}\right\}$$
$$=-\frac{dP}{d\eta}+\frac{Gr}{Re}\{\theta_r-\theta\}+\left|\frac{\partial V}{\partial \xi}\right|^{n-1}\frac{m}{m_r}$$
$$\times\left\{\frac{1}{\xi}\frac{\partial V}{\partial \xi}+n\frac{\partial^2 V}{\partial \xi^2}+\frac{1}{m/m_r}\frac{\partial(m/m_r)}{\partial \theta}\frac{\partial \theta}{\partial \xi}\frac{\partial V}{\partial \xi}\right\}. \quad (6)$$

From equations (4)–(6) the parameters of the problem are: n, Pr, Gr/Re, M, T_w/T_0.

NUMERICAL SOLUTION

The equations have been solved numerically with the aid of a CDC 7600 computer, using a modification of the implicit-explicit finite difference method of Bodoia *et al.* [13].

Momentum and energy equations are linearized, assuming that radial and axial velocity in the convective terms and $|\partial V/\partial \xi|^{n-1}$ are evaluated at the previous axial step.

The procedure of Hornbeck [14] is employed, with coarser radial mesh size in the central part of the tube ($\xi < 0.8$).

The convergence of each solution is verified changing initial and radial mesh size, the differences between the results being smaller than 2% and raising up to 3–4% for $n = 0.3$.

The mesh size normally used is $\Delta\eta = 2.5 \times 10^{-6}$ in the inlet and $\Delta\xi = 0.0125$ at the wall; for smaller n, $\Delta\eta = 2.5 \times 10^{-7}$ and $\Delta\xi = 0.00625$ are used.

COMPARISON OF THE SOLUTIONS

The average heat transfer has been studied for its interest in engineering, Nu_{av},

$$Nu_{av} = \frac{2}{\eta}\frac{1-\theta_b}{1+\theta_b}. \quad (7)$$

A detailed comparison of the present solutions with the results obtained by other workers [2, 3, 5, 6] has been performed.

For fully developed flow (constant physical properties) the comparison with Nu_{av} of Christiansen *et al.* [2, 3] and with Nu_{av} and Nu_m of Marner *et al.* [5, 6] gives discrepancies less than 2% for $n = 1$, $n = 0.5$ and 3–4% for $n = 0.3$.

The theoretical solution, obtained for forced convection and variable consistency, is compared in Fig. 1 with the analytical result of Christiansen *et al.* [2, 3]. They dropped the convective terms in the momentum equation; therefore, an infinite value of Pr number would be necessary for a proper comparison. In the present work, a large Pr value has been employed as indicated in Fig. 1. In the range $10^2 < Gz < 10^4$ the differences are smaller than $\pm 6\%$; for $Gz > 10^4$ they are greater but never exceed 10% for $n = 1$, $n = 0.5$ and 13% for $n = 0.3$. The experiments of Christiansen *et al.* [2] confirm that for $Gz > 10^4$ the values of experimental Nu_{av} are less than the analytical; in fact the value of Pr, for pseudoplastic fluids is large but always finite.

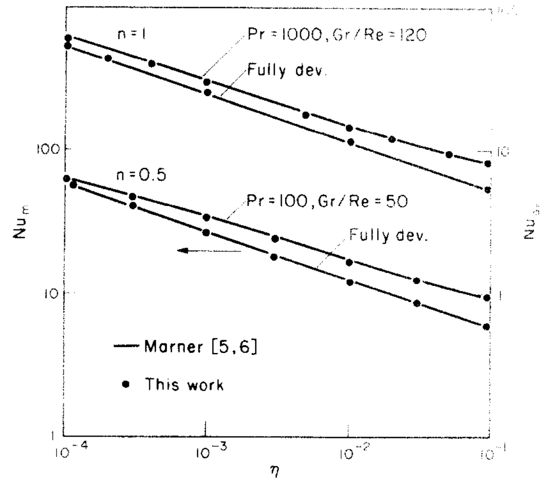

Fig. 2. Mixed convection with constant consistency.

Marner *et al.* [5, 6] used a theoretical model with constant consistency and mixed convection. Their analytical Nu_{av} and Nu_m are compared with the present results, obtained under the same conditions, and reported in Fig. 2. The agreement is very good because the present numerical method, a modification of the implicit–explicit finite difference scheme of Bodoia *et al.* [13], is very similar to that employed by Marner *et al.* [6] for mixed convection and constant consistency.

Marner *et al.* [6] made experiments in which the simultaneous variations of consistency and density can be identified. The present theoretical model contains the two combined effects. The results are comparable with those obtained experimentally by Marner *et al.* They have showed that the experimental Nu_{av} values are greater than those which they have obtained in the theoretical model with a mean deviation of 6.8% (see p. 485 of [6]). In their experiments 0.5 Carbopol solution, with an average index of pseudoplasticity of 0.87, and the following parameters: $Pr = 22.1$–43.8 and $Gr/Re = 15.1$–114.5, were employed. This work takes into account the experimental parameters used by the above authors: for consistencies and temperatures, $m_w/m_0 = 0.71$ and $T_w/T_0 = 1.1$ are assumed (see Table A1 of [6]), while $Pr = 30$ and $Gr/Re = 15$ and 65 have been used in the range of values quoted above. In Fig. 3 a difference of 5–6.5% is observed between the curves relative to constant and variable consistency. The discrepancy is 5% when $Gr/Re = 65$ (not reported in Fig. 3). These results show that most of the difference between experimental tests, with variable consistency, and theoretical solutions, with constant consistency, of Marner *et al.* [6], is eliminated with the introduction of variable consistency in the theoretical model.

This last comparison is a good piece of evidence of the correctness of the present approach when mixed convection, combined with variable consistency, are considered.

CORRELATION OF THE RESULTS

The analytical solutions, for forced convection with variable consistency, are correlated by the expression

$$Nu_{av} = 1.75\Delta w^{1/3}(m_b/m_w)^a Gz^{1/3} \quad (8)$$

where

$$a = 0.14\frac{4}{\left\{\dfrac{3n+1}{n}\right\}^n}. \quad (9)$$

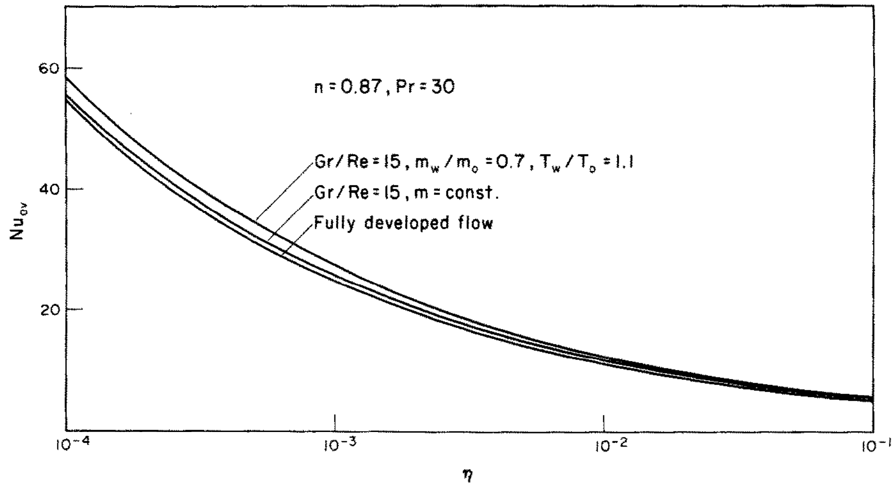

Fig. 3. Mixed convection with constant and variable consistency.

For Newtonian fluids, equation (8) corresponds to the equation introduced by Sieder and Tate [15], while Test [16] has proposed the following correlation

$$Nu_{av} = 2.1(m_b/m_w)^{0.05}(2RePrD/L)^{1/3}. \tag{10}$$

Theoretical results of Nu_{av} have been obtained for Newtonian fluids ($n = 1$) with a variation of consistency $M = -2$, $T_w/T_0 = 1.2$ and for a range of Pr values (5–1000). The Nu_{av} vs Gz curves and the results of the correlation (8) are presented in Fig. 4. For a small Pr value ($Pr = 5$) the Sieder–Tate equation gives errors less than 3.7% in the range $31.4 < Gz < 3.14 \times 10^4$. The correlation with the equation (10) of the same results ($Pr = 5$) gives discrepancies of 9% for $Gz > 1570$ while for $Gz < 1570$ the errors are greater and the maximum deviation is 17%.

A high Pr value ($Pr = 1000$) determine an increase of Nu_{av} in the inlet of the tube (for greater values of Gz) and the correlations with the equations (8) and (10) of the theoretical results, for Newtonian fluids, have shown that the errors are comparable (17%).

The equation (8) seems more convenient than the equation (10) to correlate the results for Newtonian fluids with small Pr number value.

Equation (8), with equation (9), has been employed to correlate the solutions obtained for pseudoplastic fluids and the results are presented in Fig. 4. The differences are ± 15 to ± 17%.

Pigford [17] and Christiansen *et al.* [2] have used a coefficient $a = 0.14$ for the correlation of the results of Newtonian and pseudoplastic fluids. Their theoretical solutions, obtained for an infinite Pr value, are not correlated very well in the entire range of Gz number. The results of this paper, for large Pr value, are in accordance with the observation that the exponent "a" should be a function of Gz number.

The equation (8), with equation (9), seems an improvement of the correlation used for pseudoplastic fluids.

The analytical solutions, obtained for mixed convection and constant consistency, are correlated with the expression

$$Nu_{av} = 1.75\Delta w^{1/3} \times \{Gz + 0.072(2^{3n}GrPrD/L)^{0.75}\}^{1/3} \tag{11}$$

which, for $n = 1$, reduces to the simplified form used by Martinelli and Boelter [18].

The theoretical results, for the values of Gr/Re which produce very distorted axial velocity profile, are presented in Fig. 5.

The highest differences with the results of equation (11) are ± 18%.

For a pseudoplastic fluid with a small index n and a high Pr value the maximum deviation is 18% in the intermediate Gz values but from Fig. 5 it is evident that the errors are smaller when the Pr value is 5.

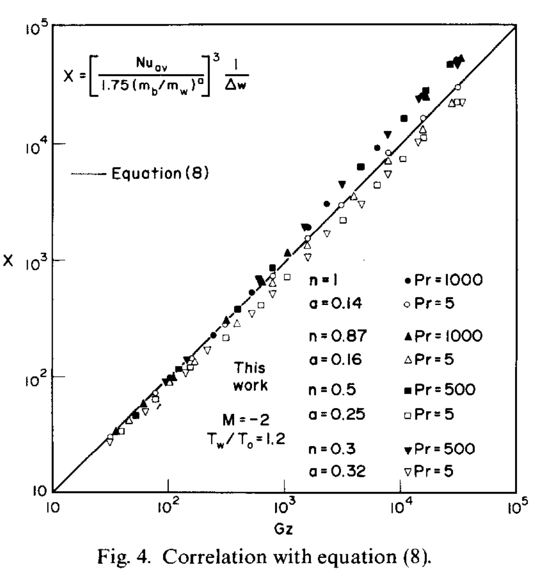

Fig. 4. Correlation with equation (8).

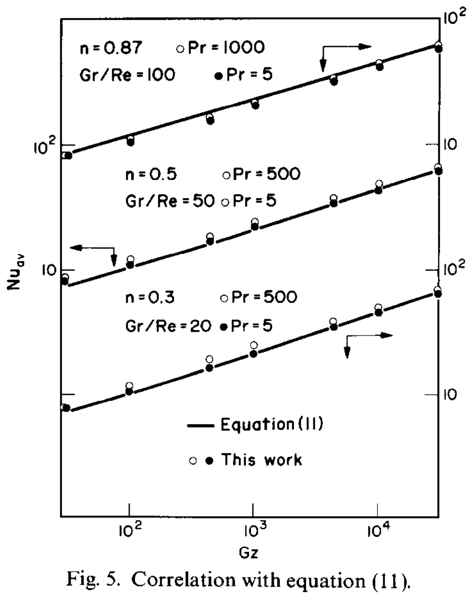

Fig. 5. Correlation with equation (11).

The equation proposed to correlate the theoretical results in the case of the combined influence of buoyancy force and consistency variation is

$$Nu_{av} = 1.75\Delta w^{1/3}\{Gz + 0.072(2^{3n}GrPrD/L)^{0.75}\}^{1/3} \times (m_b/m_w)^n. \quad (12)$$

A comparison between the results obtained using the above equation and the theoretical ones is shown in Fig. 6.

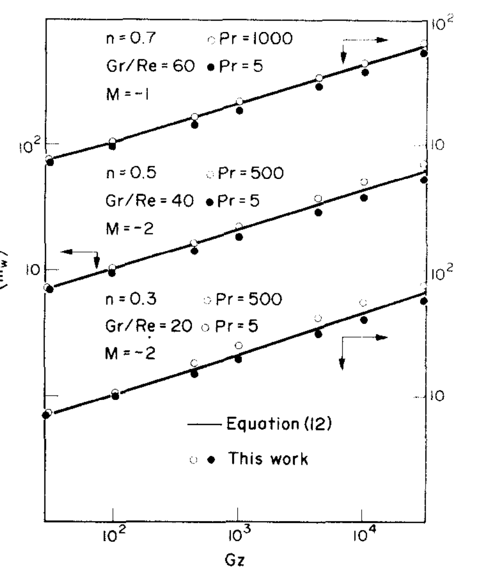

Fig. 6. Correlation with equation (12).

The greatest discrepancies are 18% for a pseudoplastic fluid with a small index n and a high Pr value in the inlet of the tube.

In conclusion, the equations (8), (11) and (12) are proposed to correlate the theoretical solutions in the heating of a pseudoplastic fluid in a vertical tube with constant wall temperature.

The ranges of the parameters Pr, Gr/Re, M are those presented in Figs. 4–6 and the maximum differences with the results obtained using the equations (8), (11), (12) are ± 15–$\pm 18\%$.

The equations proposed do not correlate the solutions of this study with a high degree of accuracy but seem interesting for they are relatively simple and applicable to a large range of the parameters of the problem.

The maximum deviation is 18% because it has been considered a range of Pr values given by 5–500 for smaller n and 5–1000 for greater n.

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